New Iterative Method to Solve Optimal Control Problems with Terminal Constraints

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Introduction

THE objective of this work is to present a new iterative method for solving optimal control problems with terminal constraints. The main idea in this work is to use a linear approximation to the state equations and a quadratic approximation to the cost functional, so that at each iteration efficient algorithms to solve linear quadratic (LQ) problems can be used to compute incremental corrections to the control signal, to make the terminal state approach the prescribed value. Many works on iterative techniques using second variations to solve optimal control problems can be found in the literature, including Refs. 1–3. The proposed scheme, however, differs from the neighboring extremal method because here the exact solution to an approximation is used in an iterative way, whereas Refs. 1–3 concern approximate solutions to the original nonlinear nonquadratic problem.

The performance of the proposed method was studied in a number of examples, including a problem of atmospheric re-entry of Shuttle vehicles.

Problem Formulation

Consider a cost functional in Lagrange form

$$J[u(\cdot)] = \int_{t_0}^{t_f} L[x(\tau), u(\tau), \tau]\, d\tau \tag{1}$$

where, for each $t \in, [t_0, t_f]$, the state $x(t) \in R^n$, the control $u(t) \in R^m$ and $L: R^n \times R^m \times R^+ \to R^+$ is assumed to be of class C^2, subject to the dynamic constraints expressed by

$$\frac{dx}{dt} = f[x(t), u(t), t] \tag{2}$$

with $f: R^n \times R^m \times R^+ \to R^n$ of class C^1 and the boundary constraints

$$x(t_0) = x_0 \qquad x(t_f) = x_f \tag{3}$$

where x_0 and x_f are fixed and given a priori.

The optimal control problem is to find, numerically, $u(t)$ for $t \in [t_0, t_f]$ so that the functional (1) is minimized subject to the constraints (2) and (3). The optimizing control is denoted $u^*(t)$ and the corresponding state $x^*(t)$.

LQ Approximation

The cost functional (1) can be expanded in a Taylor's series up to the second-order term along a nominal trajectory $x_n(\cdot)$ corresponding to a control $u_n(\cdot)$ assumed, for the moment, available:

$$J[u(\cdot)] = \int_{t_0}^{t_f} \left[\bar{L} + L_x^T \delta x + L_u^T \delta u + \frac{1}{2}\left(\delta x^T L_{xx} \delta x + 2\delta x^T L_{xu} \delta u + \delta u^T L_{uu} \delta u\right)\right] d\tau \triangleq J[u_n(\cdot)] + \delta J[\delta u(\cdot)] \tag{4}$$

where $x(t) = x_n(t) + \delta x(t), u(t) = u_n(t) + \delta u(t)$, and $\bar{L}(\tau) = L[x_n(\tau), u_n(\tau), \tau]$ whereas the state equation is linearized in the form

$$\frac{d\delta x(t)}{dt} = A(t)\delta x(t) + B(t)\delta u(t) \tag{5}$$

with

$$A(t) = \left.\frac{\partial f[x(t), u(t), t]}{\partial x(t)}\right|_{x_n(t), u_n(t)}$$
$$B(t) = \left.\frac{\partial f[x(t), u(t), t]}{\partial u(t)}\right|_{x_n(t), u_n(t)} \tag{6}$$

The nominal trajectory need not satisfy the terminal constraint $x_n(t_f) = x_f$ and, hence, the correction to be computed is required to satisfy

$$\delta x(t_f) = r_f = \alpha(x_f - x_n(t_f)) \tag{7}$$

where α is a scaling parameter.

Fact 1. The minimization of $\delta J[\delta u]$ in Eq. (4) with respect to $\delta u(\cdot)$ subject to the constraints (5) and $\delta x(t_0) = 0$ is attained by

$$\delta u^* = -L_{uu}^{-1}\left(B^T S + L_{xu}^T - B^T V P^{-1} V^T\right)\delta x^* - L_{uu}^{-1}\left(L_u + B^T V P^{-1} r_f\right) \tag{8}$$

where

$$-\dot{S} = L_{xx} - L_{xu}L_{uu}^{-1}L_{xu}^T - L_{xu}L_{uu}^{-1}B^T S + A^T S + SA - SBL_{uu}^{-1}L_{xu}^T - SBL_{uu}^{-1}B^T S$$
$$-\dot{V}\nu = L_x - L_{xu}L_{uu}^{-1}L_u + \left(-L_{xu}L_{uu}^{-1}B^T V + A^T V - SBL_{uu}^{-1}B^T V\right)\nu - SBL_{uu}^{-1}L_u \tag{9}$$
$$\nu\dot{P} = \left(V^T BL_{uu}^{-1}B^T V\right)\nu + V^T BL_{uu}^{-1}L_u$$

with $V(t_f) = I, S(t_f) = 0, P(t_f) = 0, \nu = P^{-1}(r_f - V^T\delta x)|_{t=t_0}$

Proof. Direct application of results in Bryson and Ho.[4]

Relationship Between LQ Approximation and Neighboring Extremals

Let the Hamiltonian be defined in the usual way[4]:

$$H(x, u, t, \lambda) = L(x, u, t) + \lambda^T f(x, u, t) \tag{10}$$

and consider the second variation

$$\delta J_N^2 = \int_{t_0}^{t_f} \frac{1}{2}\left(\delta x^T H_{xx}\delta x + 2\delta x^T H_{xu}\delta u + \delta u^T H_{uu}\delta u\right) dt \tag{11}$$

that appears in the neighboring extremals method.[1]

If $u_n(\cdot)$ tends to $u^*(\cdot)$, then $\delta J[\delta u]$ tends to $\delta J_N^2[\delta u]$, as seen by direct substitution of $u^*(\cdot)$ in place of $u_n(\cdot)$ in Eq. (4).

LINQUAD Method

Starting from the initial condition x_0 and using an arbitrary nominal control $u_n(\cdot)$, it is not likely that $x_n(t_f)$ will be the prescribed x_f. The idea is to correct, iteratively, the terminal constraint by computing δu such that $u_n + \delta u$, yields $x_n + \delta x$, which is closer to x_f than $x_n(t_f)$, while optimizing an approximation to the cost functional that was chosen for convenience to be of quadratic form. This is summarized in the following steps.

1) Choose an arbitrary control $u_n(\cdot)$ and scalars α and tol. [In many cases $u_n(t) = \text{const}\ \forall t$ may be adequate.]

2) Compute $x_n(t)$, $t \in [t_0, t_f]$, using Eq. (2) and $u_n(\cdot)$, starting at $x_n(t_0) = x_0$.

3) Obtain the LQ approximation computing A, B, L_x, L_u, L_{xx}, L_{xu}, and L_{uu} evaluated at $[x_n(t), u_n(t), t]$, $t \in [t_0, t_f]$.

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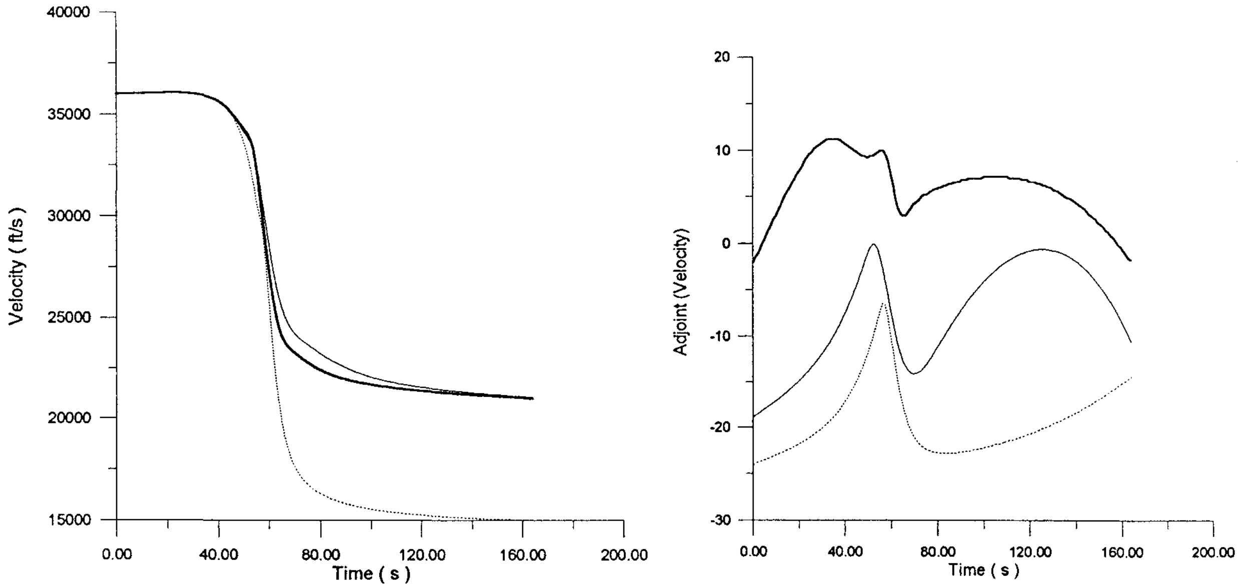

Fig. 1 Velocity and corresponding adjoint variable in Shuttle vehicle example, for the cases of nominal initial trajectory (dotted line), terminal constraint corrected by LINQUAD method (heavy solid line), and solution refined using a multiple shooting method (light solid line).

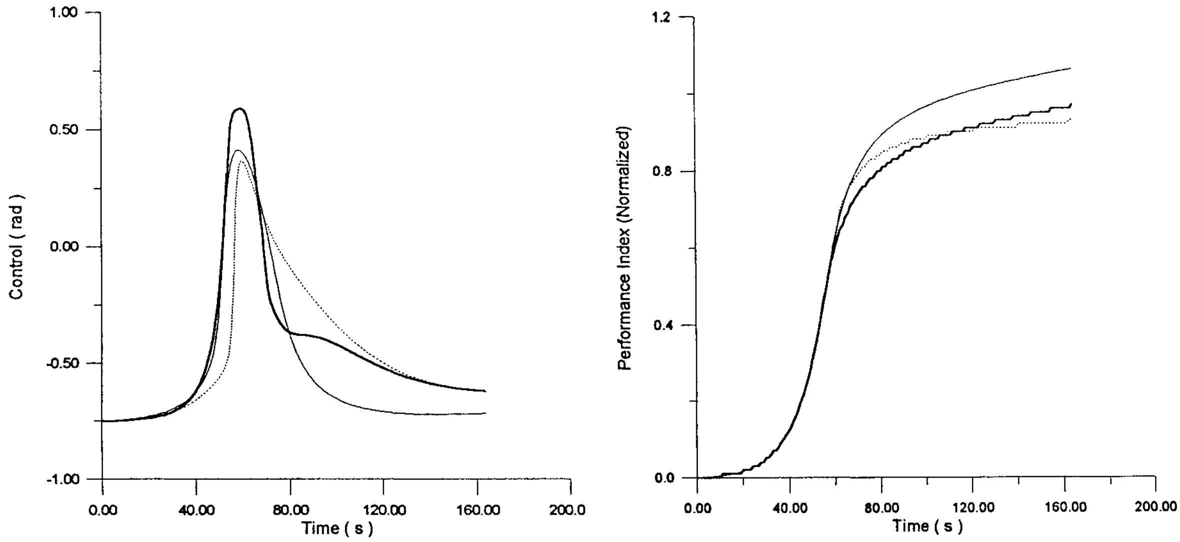

Fig. 2 Control variable and the corresponding accumulated cost, for the cases of nominal initial trajectory (dotted line), terminal constraint corrected by LINQUAD method (heavy solid line), and solution refined using a multiple shooting method (light solid line).

4) Solve the Riccati equations (9) with $r_f = \alpha[x_f - x_n(t_f)]$ and compute the incremental correction δu given by Eq. (8).

5) Let $u_{\text{new}} = u + \delta u$ and solve Eq. (2) obtaining x_{new}.

6) If $\|x_{\text{new}}(t_f) - x_f\| <$ tol, then stop; else $u_n = u_{\text{new}}$; $x_n = x_{\text{new}}$ and go to step 3.

Numerical Example: Atmospheric Re-Entry of a Shuttle Vehicle

Let R be the Earth radius and r the normalized altitude of vehicle given by $r = h/R$, where h is the altitude, v the velocity of the Shuttle vehicle, and γ the flight-path angle. The equations that describe the Shuttle are[5,6]

$$\frac{dr}{dt} = \frac{v \sin\gamma}{R}$$

$$\frac{dv}{dt} = -\frac{S_a \rho v^2 c_D(u)}{2m} - \frac{g \sin\gamma}{(1+r)^2} \tag{12}$$

$$\frac{d\gamma}{dt} = \frac{S_a \rho v c_L(u)}{2m} + \frac{v\cos\gamma}{R(1+r)} - \frac{g\cos\gamma}{v(1+r)^2}$$

where $\rho = \rho_0 e^{-\beta R r}$ is the atmospheric density, $c_D(u) = c_{D0} + c_{D1}\cos(u)$ is the aerodynamical drag coefficient, $c_L(u) = c_{L0}\sin(u)$ is the aerodynamical lift coefficient, u is the attack angle, g is the gravitational acceleration, and S/m is a constant (frontal area)/(mass of vehicle).

The chosen cost functional is related to the heating of the vehicle in the Earth's atmosphere:

$$J[u(\cdot)] = \int_{t_0}^{t_f} \frac{10 S v^3 \sqrt{\rho} c_L(u)}{m} \, dt \tag{13}$$

where $t_0 = 0$, and $t_f = 164$ s.

The simulations were carried out using the numerical values from Bulirsh et al.[5] and the boundary conditions: $h(0) = 400.000$ ft, $v(0) = 36.000$ ft/s, $\gamma(0) = -8.1 \deg \pi/180$ rad, $r(t_f) = 250.000$ ft, $v(t_f) = 18{,}630$ ft/s, and $\gamma(t_f) = 0$ rad.

To test the proposed method, a solution u^{approx} was found to the problem with the same dynamics and cost functional, but with a different set of terminal constraints. Then the linear quadratic method (LINQUAD) was applied to find the optimal solution for the correct set of terminal constraints using u^{approx} as the initial guess for u_n. Figure 1 shows the comparison between the optimal trajectory found by the LINQUAD method, later refined using a multiple shooting method. Also shown is the behavior of the adjoint variables. Figure 2 shows the control signal and the attained cost for the nominal, LINQUAD, and multiple shooting solutions.

Conclusions

The LINQUAD method can be used to provide fine adjustments to the terminal constraints or to generate starting trajectories for the adjoint process in the multiple shooting method in order to refine an existing approximate solution to an optimal control problem with fixed endpoints. In general terms, the choice of the initial nominal trajectory was found to be crucial for successful convergence and is related to the fact that the quadratic approximation of the cost functional may not be adequate in a region too far from the optimal trajectory.

It was also noted that LINQUAD is very closely related to the neighboring extremals method. The LINQUAD method is very easily implemented because one can take advantage of the existing linear quadratic regulator algorithms.

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